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ON THE STABILITY OF AXISYMMETRICAL FLOWS

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SPACE SCIENCES LABORATORY

GENERAL  ELECTRIC

MISSILE AND SPACE DIVISION

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# SPACE SCIENCES LABORATORY

AEROPHYSICS OPERATION

ON THE STABILITY OF AXISYMMETRICAL FLOWS\*

By

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## SECTION 1 - GENERAL THEORY

Because of the intercontinental ballistic missile and satellite programs, the hypersonic flow around a body of revolution has been investigated. Recently the wake of such a body in a hypersonic flow becomes one of the most interesting problems among aerodynamicists due to the desire to monitor the re-entry bodies. The thermal radiation emitted by the hot gas left in the trail and the electron density in the trail are the observables in the trails of such bodies. In order to determine the observables, we have to know, in detail, the flow field of the wakes behind the body, which depends mainly on the flow in the boundary layer of the body. The flow field in the boundary layer of the body as well as that in the wake may be either laminar or turbulent. The properties of the observables depend greatly on whether the flow in the wake is laminar or turbulent. Hence it is extremely interesting to know the transition point or the transition region of the flow field on and behind a body of revolution. Even though there is no theoretical analysis available to predict accurately the transition point, the stability of a laminar flow has a close relation with the transition. In this report, we will study the stability of axisymmetrical flow in general, which includes both the stability of axisymmetrical wake or jet and of boundary layer flow on a slender body of revolution.

A detailed analysis of the stability of a hypersonic flow over a body is very complicated because not only is the flow compressible, but the composition of the air may vary considerably from place to place. An accurate analysis should include both the effect of compressibility and the effect of the composition of the gas in the flow field. However, since the stability of the axisymmetrical flow of an incompressible fluid has not yet been well developed<sup>1</sup>, it would be useful to investigate this case first to bring out the essential points of stability of an axisymmetrical flow before an extensive program of the stability of an axisymmetrical flow of a compressible gas with the influence of variable composition is carried out.

The problem is formulated in a general manner, so that the results may be applied to both flow problems with solid boundaries, such as boundary layer flows and pipe flows, and those without solid boundaries, such as wake flows and jet flows.

Since we consider only the axisymmetrical basic flow, the general disturbances will be rotationally symmetric disturbances. We shall formulate the problem of the stability of an axisymmetric flow subjected to a rotationally symmetric disturbance in Section 2. But we shall show in Section 3 that the axisymmetrical flow is always stable with respect to disturbance of the tangential velocity component. As a result we need only to study the stability with respect to axisymmetrical disturbances in detail in Sections 4 to 6.

The stability equation of axisymmetrical disturbances is investigated and its solution for large Reynolds number is found for arbitrary basic velocity profile in Section 4. The boundary value problems of the stability for various axisymmetrical flows are analysed in Section 5. However, the detailed numerical calculations will not be given here but in later parts of this report. The comparison of the stability of axisymmetrical flow with that of two-dimensional flow will be discussed according to an equivalent principle in Section 6.

## SECTION 2 - GENERAL FORMULATION OF THE PROBLEM

We use cylindrical coordinates  $r, \theta, z$  with velocity components  $u, v, w$  in the direction of  $r, \theta, z$ , respectively. For an incompressible and viscous fluid, the equation of continuity and equations of motion in the cylindrical coordinates are, respectively,

$$\frac{1}{r} \frac{\partial ru}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \quad (2.1)$$

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \\ \nu \left( \nabla^2 u - \frac{u}{r^2} + \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right) \end{aligned} \quad (2.2)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \\ \nu \left( \nabla^2 v + \frac{2}{r^2} \frac{\partial u}{\partial \theta} - \frac{v}{r} \right) \end{aligned} \quad (2.3)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 w \quad (2.4)$$

where 
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

For an axisymmetrical flow subjected to rotationally symmetric disturbances, all the variables are independent of the angular displacement. Hence, equations (2.1) to (2.4) reduce to

$$\frac{1}{r} \frac{\partial ru}{\partial r} + \frac{\partial w}{\partial z} = 0 \quad (2.5)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu (\nabla_1^2 u - \frac{u}{r^2}) \quad (2.6)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = \nu (\nabla_1^2 v - \frac{v}{r^2}) \quad (2.7)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla_1^2 w \quad (2.8)$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

$\rho$  is the density,  $p$  is the pressure and  $\nu$  is the coefficient of kinematic viscosity of the fluid. Equations (2.5) to (2.8) are the basic equations for our stability analysis.

We consider only the stability of parallel or nearly parallel flows which include flows of the types of boundary layer, wake, and jet. The basic velocity profile of these parallel and nearly parallel flows may be considered as follows:

$$U = V = 0, \quad W = W(r) \quad (2.9)$$

From now on, all quantities are to be non-dimensional such that all velocities are expressed in terms of a characteristic velocity  $U_0(z)$ , all lengths are expressed in terms of a characteristic length  $L(z)$ , the pressure is expressed in terms of  $\rho V_0^2$ , the time is expressed in terms of  $L/U_0$  and the Reynolds number of our problem is

$$R = \frac{L U_0}{\nu} \quad (2.10)$$



For nearly parallel flow,  $U_0$  and  $L$  are usually functions of the axial distance  $z$ . We shall discuss the choice of proper values for  $U_0$  and  $L$  when we investigate specific flow problems in the later parts of this report.

For a rotationally symmetrical disturbance, all the variables are independent of the angular displacement  $\theta$ . Hence, the velocity components and the pressure may be written in the following form:

$$\begin{aligned} u(r, z, t) &= u(r) \exp [i\alpha (z - ct)] \\ v(r, z, t) &= v(r) \exp [i\alpha (z - ct)] \\ w(r, z, t) &= W(r) + w(r) \exp [i\alpha (z - ct)] \\ p(r, z, t) &= P(z) + p(r) \exp [i\alpha (z - ct)] \end{aligned} \tag{2.11}$$

Where  $W(r)$  and  $P(z)$  are respectively the velocity and pressure of the basic flow;  $u = u(r)$ ,  $v = v(r)$ ,  $w = w(r)$  and  $p = p(r)$  are respectively the perturbed velocity components and pressure, which are functions of  $r$  only;  $\alpha$  is the wave number which is real and positive and  $c$  is the complex wave velocity, i.e.,

$$c = c_r + i c_i \tag{2.12}$$

with  $i = \sqrt{-1}$

Substituting equation (2.11) into equations (2.5) to (2.8) and neglecting the higher order terms of the perturbed quantities, we obtain the following equations for the perturbed quantities

$$(ru)' + i\alpha r w = 0 \tag{2.13}$$

$$i \alpha (W - c) u = -p' + \frac{1}{R} \left( u'' + \frac{w}{r} - \alpha^2 u - \frac{u}{2} \right) \quad (2.14)$$

$$i \alpha (W - c) w + u W' = -i \alpha p + \frac{1}{R} \left( w'' + \frac{w'}{r} - \alpha^2 w \right) \quad (2.15)$$

$$i \alpha (W - c) r = \frac{1}{R} \left( v'' + \frac{v'}{r} - \alpha^2 r - \frac{r}{2} \right) \quad (2.16)$$

where prime refers to the differentiation with respect to  $r$ .

It is interesting to notice that equations (2.13) to (2.16) may be divided into two independent groups:

- (i) Equation (2.16) is independent of the other three equations. Since equation (2.16) consists of the unknown  $v(r)$  only, this equation may be used to study the stability of an axisymmetrical flow to the tangential velocity disturbances.
- (ii) Equations (2.13) to (2.15) are coupled. These three equations may be used to study the stability of an axisymmetrical flow due to some axisymmetrical disturbances  $u$ ,  $w$  and  $p$ .

We may reduce equations (2.13) to (2.15) into a single differential equation for the function

$$f = r u \quad (2.17)$$

It is easy to show that the function  $f$  is proportional to the perturbed stream function of the axisymmetrical flow.

Eliminating the pressure  $p(r)$  from equations (2.14) and (2.15) we have

$$\alpha^2 (W - c) u + i \alpha (W - c) w' + i \alpha W' w + u' W' + u W'' = \frac{1}{R} \left( w''' + \frac{w''}{r} - \frac{w'}{r^2} - \alpha^2 w' \right) - \frac{i \alpha}{R} \left( u'' + \frac{w}{r} - \alpha^2 u - \frac{u}{2} \right) \quad (2.18)$$

We may express  $u$ ,  $w$  and their derivatives with respect to  $r$  in terms of  $f$  and its derivatives in  $r$  by the help of equations (2.13) and (2.17). As a result, equation (2.18) gives

$$(W - c) (f''' - \alpha^2 f - \frac{f'}{r}) - f (W'' - \frac{W'}{r}) = - \frac{i}{\alpha R}$$

$$\left[ (f^{iv} - 2\alpha^2 f'' + \alpha^4 f) - \frac{2}{r} (f''' - \alpha^2 f') + \frac{3}{r^2} (f'' - \frac{f'}{r}) \right] \quad (2.19)$$

Equation (2.19) is the fundamental equation for the stability of an axisymmetrical flow subjected to axisymmetrical disturbances. It is interesting to notice that if those terms containing explicitly the factor  $r$  vanish, equation (2.19) reduces to the well known Orr-Sommerfeld equation for the stability of two-dimensional parallel flows.<sup>2</sup>

For the flow field between  $r = r_1$  and  $r = r_2$ , equation (2.19) is then to be solved under the boundary conditions

$$f(r_1) = 0, f'(r_1) = 0, f(r_2) = 0, f'(r_2) = 0 \quad (2.20)$$

Since equation (2.19) is a fourth order ordinary differential equation, there exists a fundamental system of four solutions of equation (2.19) which are analytic functions of the variable  $r$  and of the parameter  $c$ ,  $\alpha$  and  $\alpha R$ . Let the four solutions be  $f_1$ ,  $f_2$ ,  $f_3$ , and  $f_4$ . The conditions (2.20) gives the determinantal equation

$$F(c, \alpha, \alpha R) = \begin{vmatrix} f_1(r_1) & f_2(r_1) & f_3(r_1) & f_4(r_1) \\ f_1(r_2) & f_2(r_2) & f_3(r_2) & f_4(r_2) \\ f'_1(r_1) & f'_2(r_1) & f'_3(r_1) & f'_4(r_1) \\ f'_1(r_2) & f'_2(r_2) & f'_3(r_2) & f'_4(r_2) \end{vmatrix} = 0 \quad (2.21)$$

Since the function  $F(c, \alpha, \alpha R)$  is an entire function of the variables  $c$ ,  $\alpha$ , and  $\alpha R$ , we may solve for  $c$  and obtain

$$c = c(\alpha, R) \quad (2.22)$$

Since  $\alpha$  and  $R$  are taken to be real and positive, we have four equations (2.21)

$$\text{and (2.22) } c_r = c_r(\alpha, R); c_i = c_i(\alpha, R) \quad (2.23)$$

The curve  $c_i = 0$  gives the limit of stability.

The general solutions of equation (2.19) for arbitrary basic velocity profile  $W(r)$  are difficult to obtain. In Section 4, we shall find the asymptotic solutions for the limiting cases of very large  $\alpha R$ .

### SECTION 3 - STABILITY OF AN AXISYMMETRICAL FLOW WITH RESPECT TO THE DISTURBANCES OF THE TANGENTIAL VELOCITY COMPONENT V

The function  $v$  of the perturbed tangential velocity component of equation (2.16) should be considered as a complex function of  $r$ . If we multiply equation (2.16) by  $r\bar{v}$  where bar denotes the complex conjugate, and integrate over the interval  $(r_1, r_2)$  with respect to  $r$ , we obtain

$$i \alpha \int_{r_1}^{r_2} (W - c) v \bar{v} r dr = \frac{1}{R} \int_{r_1}^{r_2} \left( v'' + \frac{v'}{r} - \alpha^2 v - \frac{v}{r^2} \right) \bar{v} r dr \quad (3.1)$$

but

$$\begin{aligned} \int_{r_1}^{r_2} v'' \bar{v} r dr &= v' \bar{v} r \Big|_{r_1}^{r_2} - \int_{r_1}^{r_2} (v' \bar{v} + v' \bar{v}' r) dr = \\ &- \int_{r_1}^{r_2} (v' \bar{v} + v' \bar{v}' r) dr \end{aligned} \quad (3.2)$$

where the boundary conditions  $v'(r_1) = v'(r_2) = 0$  are used.

Substituting equations (2.12) and (3.2) into equation (3.1) we have

$$\begin{aligned} i \alpha \left\{ \int_{r_1}^{r_2} (W - c_r) v \bar{v} r dr \right\} + \alpha c_i \int_{r_1}^{r_2} v \bar{v} r dr \\ = - \frac{1}{R} \int_{r_1}^{r_2} \left( v' \bar{v}' r + \alpha^2 v \bar{v} r + \frac{v \bar{v}}{r} \right) dr \end{aligned} \quad (3.3)$$

From the imaginary part of equation (3.3) we have

$$c_r = \frac{\int_{r_1}^{r_2} W v \bar{v} r dr}{\int_{r_1}^{r_2} v \bar{v} r dr} \quad (3.4)$$

Equation (3.4) gives the velocity of propagation of this tangential velocity disturbance.

From the real part of equation (3.3), we have

$$c_i = -\frac{1}{\alpha R} \cdot \frac{\int_{r_1}^{r_2} (v' \bar{v}' r + \alpha^2 v \bar{v} r + \frac{v \bar{v}}{r}) dr}{\int_{r_1}^{r_2} v \bar{v} r dr} \quad (3.5)$$

Since the integrals in equation (3.5) are all positive quantities,  $c_i$  is then always negative. As a result, the tangential velocity disturbances will always be damped out for all axisymmetrical flows subjected to rotationally symmetric disturbances. It was Lew<sup>3</sup> who first showed that this result holds true for axisymmetrical jet flow. Since the flow is stable with respect to  $v$ , we need only to consider the axisymmetric disturbances governed by equation (2.19)

SECTION 4 - APPROXIMATE SOLUTIONS OF THE STABILITY EQUATION OF  
THE AXISYMMETRIC FLOW (2.19)

We introduce a new variable  $\xi$  such that

$$\xi = 1 - \frac{r^2}{r_c^2} \quad (4.1)$$

where  $r = r_c$  when  $W = c$ .

Equation (2.19) in terms of  $\xi$  becomes

$$(W - c) Lf - (1 - \xi) W''f = -\frac{i}{\alpha_1 R_1} LLf \quad (4.2)$$

where prime refers to derivative with respect to  $\xi$  and

$$f(r) = f(\xi)$$

$$L = (1 - \xi) \frac{d^2}{d\xi^2} - \alpha_1^2$$

$$\alpha_1 = \frac{r_c}{2} \alpha, \quad R_1 = \frac{r_c}{2} R \quad (4.3)$$

It is interesting to notice the similarity of equation (4.2) with the Orr-Sommerfeld equation of two-dimensional parallel flow. If the factor  $(1 - \xi)$  of equation (4.2) is replaced by unity, equation (4.2) reduces exactly to the Orr-Sommerfeld equation of the stability of two-dimensional flow. Because of the similarity, we may obtain the solutions of equation (4.2) by the same method used in the analysis of Orr-Sommerfeld equations. The mathematical properties of the solutions of equation (4.2) are similar to those of Orr-Sommerfeld equations.<sup>2</sup>

There exists a fundamental system of four solutions of equation (4.2) which are analytic functions of the variable  $\xi$  and of the parameter  $c$ ,  $\alpha_1$  and  $\alpha_1 R_1$ , being, in fact, entire functions of these parameters.

We may find the solutions of equation (4.2) by convergent series in terms of a small parameter  $\epsilon$ . First we make a change of variable

$$\xi = \epsilon \eta, \quad f(\xi) = \phi(\eta) \quad (4.4)$$

Equation (4.2) becomes

$$\begin{aligned} (W - c) \left[ (1 - \epsilon \eta) \frac{d^2 \phi}{d\eta^2} - \alpha_1^2 \epsilon^2 \phi \right] - (1 - \epsilon \eta) \epsilon^2 \phi W'' = \\ - \frac{i}{\alpha_1 R_1 \epsilon^2} \left[ (1 - \epsilon \eta)^2 \frac{d^4 \phi}{d\eta^4} - 2 \alpha_1^2 \epsilon^2 (1 - \epsilon \eta) \frac{d^2 \phi}{d\eta^2} + \right. \\ \left. \alpha_1^4 \epsilon^4 \phi - 2 (1 - \epsilon \eta) \epsilon \frac{d^3 \phi}{d\eta^3} \right] \end{aligned} \quad (4.5)$$

where

$$\left. \begin{aligned} W - c &= W'_0(\epsilon \eta) + \frac{W''_0}{2!} (\epsilon \eta)^2 + \dots \\ W'' &= W''_0 + W'''_0(\epsilon \eta) + \frac{W^{iv}_0}{2!} (\epsilon \eta)^2 + \dots \end{aligned} \right\} \quad (4.6)$$

subscript 0 refers to value at  $\xi = 0$ . The solution is then obtained in the form

$$f(\xi) = \phi(\eta) = \phi^{(0)}(\eta) + \epsilon \phi^{(1)}(\eta) + \epsilon^2 \phi^{(2)}(\eta) + \dots \quad (4.7)$$



and the differential equations for the approximations of successive orders can be obtained by substituting equations (4.6) and (4.7) into (4.5) and equating all the coefficients of the various powers of  $\epsilon$  to zero. From equation (4.5) we find that the proper choice of the parameter  $\epsilon$  is

$$\epsilon = (\alpha_1 R_1)^{-\frac{1}{3}} \quad (4.8)$$

The differential equations for the functions  $\phi^{(0)}(\eta)$ ,  $\phi^{(1)}(\eta)$ , etc., are as follows:

$$\left. \begin{aligned} \epsilon^0 &= W'_0 \eta \phi^{(0)''} + i \phi^{(0)iv} = 0 \\ \epsilon^n &= W'_0 \eta \phi^{(n)iv} = L_{(n-1)}(\phi), \quad (n \geq 1) \end{aligned} \right\} \quad (4.9)$$

where  $L_{(n-1)}(\phi)$  is a linear combination of  $\phi^{(0)}(\eta)$ ,  $\phi^{(1)}(\eta)$ , . . .  $\phi^{(n-1)}(\eta)$  and their derivatives. In particular

$$L_0(\phi) = (W'_0 - \frac{W''_0}{2}) \eta^2 \phi_0 + W''_0 \phi_0 + 2i(\eta \phi^{(0)iv} + \phi^{(0)''''}) \quad (4.10)$$

It is interesting to notice that the zeroth order solutions  $\phi^{(6)}_\eta$  are exactly the same as those of the two-dimensional case,<sup>2</sup> i.e.,

$$\left. \begin{aligned} \phi_1^{(0)} &= \eta ; \phi_3^{(0)} = \int_{+\infty}^{\eta} d\eta \int_{+\infty}^{\eta} d\eta \cdot \eta^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}(z) \\ \phi_2^{(0)} &= 1 ; \phi_4^{(0)} = \int_{-\infty}^{\eta} d\eta \int_{-\infty}^{\eta} d\eta \cdot \eta^{\frac{1}{2}} H_{\frac{1}{3}}^{(2)}(z) \end{aligned} \right\} \quad (4.11)$$

where  $z = \frac{2}{3} \left[ i (W')^{+1/3} \eta \right]^{3/2}$ ,  $H_{1/3}^{(1)}$  and  $H_{1/3}^{(2)}$  are, respectively, the first and the second kind of Hankel function of the order  $1/3$ .

The solutions of the higher order terms  $\phi^{(n)}(\eta)$ ,  $n \geq 1$  of equation (4.9) are different from those of the two-dimensional case because the functions  $L_{(n-1)}(\phi)$  are different from the corresponding functions in the two-dimensional case. The higher order terms are given by

$$\phi_i^{(n)} = \frac{\pi}{6} \int_0^\eta d\eta \int_0^\eta d\eta \left\{ \phi_4^{(0)''} \int_0^\eta d\eta \phi_3^{(0)} L_{(n-1)}(\phi) - \phi_3^{(0)''} \int_0^\eta d\eta \cdot \phi_4^{(0)''} L_{(n-1)}(\phi) \right\} \quad (4.12)$$

The convergent series (4.7) is convergent provided  $\epsilon$  is restricted so that the series (4.6) are convergent.

From the study of the stability of two-dimensional flow, we know that it is usually more convenient to use asymptotic series for numerical purposes, particularly in dealing with boundary value problems. It was found that the most convenient way is to replace the first two solutions  $\phi_1^{(0)}$  and  $\phi_2^{(0)}$  by the corresponding asymptotic solutions  $f_1^{(0)}$  and  $f_2^{(0)}$ . We may approximate the four fundamental solutions  $\{f_1, f_2, f_3, f_4\}$  by the functions  $\{f_1^{(0)}, f_2^{(0)}, \phi_3^{(0)}, \phi_4^{(0)}\}$  in the boundary value problems. The asymptotic solutions of  $f_1$  and  $f_2$  may be obtained by developing  $f(\xi)$  in powers of  $(\alpha_1 R_1)^{-1}$ . We put

$$f(\xi) = f^{(0)}(\xi) + (\alpha_1 R_1)^{-1} f^{(1)}(\xi) + \dots \quad (4.13)$$

Substituting equation (4.13) into equation (4.2) and collecting terms of the same powers of  $(\alpha_1 R_1)^{-1}$ , we have

$$(W - c) \left[ (1 - \xi) f^{(0)''} - \alpha_1^2 f^{(0)} \right] - (1 - \xi) W'' f^{(0)} = 0 \quad (4.14)$$

$$\begin{aligned} (W - c) \left[ (1 - \xi) f^{(n)''} - \alpha_1^2 f^{(n)} - (1 - \xi) W'' f^{(n)} \right] = \\ - i \left[ (1 - \xi)^2 f^{(n-1)iv} - 2\alpha_1^2 (1 - \xi) f^{(n-1)''} + \alpha_1^4 f^{(n-1)} - 2(1 - \xi) \right. \\ \left. f^{(n-1)'''} \right] \quad n \geq 1 \end{aligned} \quad (4.15)$$

The equations of the asymptotic solutions (4.14) and (4.15) are similar to those of the two-dimensional case, but are not exactly the same even for the zeroth order term  $f^{(0)}$ , because  $f^{(0)}$  corresponds to the first two terms of the convergent series (4.7), i. e.,  $f^{(0)} \cong \phi^{(0)} + \epsilon \phi^{(1)}$ . Since  $\phi^{(1)}$  is different from the corresponding function in the two-dimensional case,  $f^{(0)}$  is different from the corresponding function of the two-dimensional case too.

There are two solutions of the type of (4.13), which are known as inviscid solutions, and which can be obtained by developing  $f^{(0)}$  in powers of  $\alpha_1^2$ . The two particular integrals are

$$f_1^{(0)} = (W - c) \left[ h_0(\xi) + \alpha_1^2 h_2(\xi) + \alpha_1^4 h_4(\xi) + \dots \right] \quad (4.16)$$

$$f_2^{(0)} = (W - c) \left[ k_1(\xi) + \alpha_1^2 k_3(\xi) + \alpha_1^4 k_5(\xi) + \dots \right] \quad (4.17)$$

where

$$h_0(\xi) = 1$$

$$h_{2n+2} = \int_{\xi_a}^{\xi} \frac{1}{(W-c)^2} \left[ \int_{\xi_a}^{\xi} \frac{h_{2n}(W-c)^2}{(1-\xi)} d\xi \right] d\xi \quad (4.18)$$

$$n \geq 0$$

$$k_1(\xi) = \int_{\xi_a}^{\xi} \frac{d\xi}{(W-c)^2}$$

$$k_{2n+3}(\xi) = \int_{\xi_a}^{\xi} \frac{1}{(W-c)^2} \left[ \int_{\xi_a}^{\xi} \frac{k_{2n+1}(W-c)^2}{(1-\xi)} d\xi \right] d\xi \quad (4.19)$$

$$n \geq 0$$

where  $\xi_a$  may be any fixed point but  $\xi_a \neq 1$ .

Near the axis of symmetry  $\xi \approx 1$ , we may approximate the velocity profile in the following form:

$$W(r) = W_1 - B r^{2n} \quad (4.20)$$

where  $W_1$ ,  $B$  and  $n$  are constants to be determined from the actual velocity profile. Hence,

$$\left. \begin{aligned} W(\xi) - c &= B r_c^{2n} \left[ 1 - (1-\xi)^n \right] \\ W'' &= -B r_c^{2n} n \left[ (n-1)(1-\xi)^{n-2} \right] \end{aligned} \right\} \quad (4.21)$$

Equation (4.14) for the velocity profile (4.20) becomes

$$\left[ 1 - (1 - \xi)^n \right] \left[ (1 - \xi) \frac{d^2 \psi}{d \xi^2} - \alpha_1^2 \psi \right] + n(n-1)(1 - \xi)^{n-1} \psi = 0 \quad (4.22)$$

where  $\psi = f$  for the velocity profile (4.20). Equation (4.22) is the Pretsch's equation<sup>1</sup> which has two solutions

$$\begin{aligned} f_1^{(0)} &= \psi_1(\xi) = e_0 \xi P_1(\xi) \\ f_2^{(0)} &= \psi_2(\xi) = \left[ P_2(\xi) + A \psi_1 \ln \xi \right] + K \psi_1 \end{aligned} \quad (4.23)$$

for  $\xi_a \leq \xi \leq 1$

where  $P_1(\xi)$ ,  $P_2(\xi)$  and  $A$  for  $n = 1$  to  $5$  are given in Reference 1.  $P_1(\xi)$  and  $P_2(\xi)$  are power series of  $\xi$ . The constants  $e_0$  and  $K$  are determined by joining the solutions  $f_i^{(0)}$  and  $\psi_i^{(0)}$  at the fixed point  $\xi = \xi_a$ , i.e.,

$$\begin{aligned} \psi_1(\xi_a) &= e_0 \xi_a P_1(\xi_a) = f_1^{(0)}(\xi_a) = W(\xi_a) - c \\ \psi_2(\xi_a) &= \frac{1}{e_0} \left[ P_2(\xi_a) + \frac{A}{e_0} \psi_1(\xi_a) \ln \xi_a \right] + K \psi_1(\xi_a) = \\ f_2^{(0)}(\xi_a) &= 0 \end{aligned} \quad (4.24)$$

We shall use the functions  $f_i^{(0)}$  of equations (4.16) and (4.17) for  $\xi_a \geq \xi \geq -\infty$  and the functions  $\psi_i$  of equation (4.23) for  $1 \geq \xi \geq \xi_a$  for the inviscid solutions  $f_1$  and  $f_2$ , respectively

Having found the two particular integrals of  $f^{(0)}$ , we can obtain the higher approximations by quadratures. However, in the calculation of stability of an axisymmetrical flow, it is sufficient to use

$$\left\{ f_1^{(0)}, f_2^{(0)}, \phi_3^{(0)}, \phi_4^{(0)} \right\}.$$

We may obtain the viscous solutions of the asymptotic series in the same manner as in the two-dimensional case. Since we are not going to use these asymptotic viscous solutions, we shall not give them here.

## SECTION 5 - THE BOUNDARY VALUE PROBLEM

Having obtained the fundamental solutions, we may solve the eigenvalue problems for any parallel or nearly parallel axisymmetrical flow. Because of some simplifications, we may divide the boundary value problems into two different classes as follows:

- (i) Flow within a finite domain. In this case both the end points  $\xi_1$  and  $\xi_2$  are finite. One of the two limits may be the axis of symmetry, i.e.,  $r = 0$ , or  $\xi = 1$ . The boundary conditions are those given in equation (2.20) which may be expressed in terms of  $\xi$  as

$$f(\xi_1) = f'(\xi_1) = f(\xi_2) = f'(\xi_2) = 0 \quad (5.1)$$

The factor  $\sqrt{1 - \xi}$  associated with  $f' = \frac{df}{d\xi}$  will be dropped out in the determinantal equation. Hence, we may use the boundary conditions in the form of equation (5.1). It should be noticed that on the axis of symmetry, the boundary conditions are still

$$f(1) = f'(1) = 0$$

because the limiting conditions that the radial velocity component on the axis of symmetry vanishes, give these two conditions. This is one of the essential differences of the axisymmetrical problems from those of symmetrical flow problem of the two-dimensional case.

For the latter case, only one of the conditions  $f(0)$  or  $f'(0)$  will vanish depending whether the disturbance is symmetrical or antisymmetrical.

The determinantal equation corresponding to the boundary conditions (5.1) is

$$F_1(c, \alpha_1, \alpha_1 R) = \begin{vmatrix} f_{11} & f_{21} & f_{31} & f_{41} \\ f_{12} & f_{22} & f_{32} & f_{42} \\ f'_{11} & f'_{21} & f'_{31} & f'_{41} \\ f'_{12} & f'_{22} & f'_{32} & f'_{42} \end{vmatrix} = 0 \quad (5.2)$$

where

$$f_{11} = f_1(\xi_1), \quad f'_{11} = f'_1(\xi_1), \text{ etc.}$$

Similar to the two-dimensional case,<sup>2</sup> if we neglect the terms of the order of  $(\alpha_1 R_1)^{-1}$  and

$$\text{exponent} \quad \left\{ - \int_{\xi_1}^{\xi_2} \sqrt{i \alpha R (W - c)} \, d\xi \right\}$$

equation (5.2) becomes

$$\frac{G_1(\alpha_1, c)}{G_3(\alpha_1, c)} = \frac{f_{31}}{f'_{31}} + \frac{G_2(\alpha_1, c)}{G_4(\alpha_1, c)} \frac{f_{42}}{f'_{42}} \quad (5.3)$$

where

$$\begin{aligned} G_1 &= \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} & G_2 &= \begin{vmatrix} f_{11} & f_{21} \\ f'_{12} & f'_{22} \end{vmatrix} \\ G_3 &= \begin{vmatrix} f'_{11} & f'_{12} \\ f'_{21} & f'_{22} \end{vmatrix} & G_4 &= \begin{vmatrix} f'_{11} & f'_{21} \\ f'_{12} & f'_{22} \end{vmatrix} \end{aligned} \quad (5.4)$$



Equation (5.4) is exactly the same as that of the corresponding problem for two-dimensional flow between solid walls in relative motions. \*

- (ii) Flow with an infinite domain. It is the case for wake, jet, or boundary layer flow on the outside of a circular cylinder or a body of revolution. In this case, one of the end points  $\xi_1$  or  $\xi_2$  will be minus infinity, i.e.,  $r_2 = \infty$ ,  $\xi_2 = -\infty$ . Hence, one of the viscous solutions  $\phi_3^{(0)}(-\infty)$  is infinitely large. Our boundary conditions require that  $f$  should be a linear combination of  $f_1^{(0)}$ ,  $f_2^{(0)}$  and  $\phi_4^{(0)}$  alone, i.e.,

$$f = c_1 f_1^{(0)} + c_2 f_2^{(0)} + c_4 \phi_4^{(0)} \quad (5.5)$$

The boundary conditions at the finite limit point  $\xi_1 \neq \infty$  are still

$$f(\xi_1) = \sqrt{1 - \xi_1} \quad f'(\xi_1) = 0 \quad (5.6a)$$

At  $\xi_2 = -\infty$ , the fundamental solution  $\phi_4^{(0)}$  tends to be zero very rapidly. The two inviscid solutions behave like  $e^{\pm \alpha r}$ . The condition that  $f$  tends to be zero as  $r$  tends to be infinite excludes the integral  $e^{+\alpha r}$ . Hence  $f$  must be proportional to  $e^{-\alpha r}$  for  $r > r_b$ . This condition may be expressed in terms of  $\xi$  as follows:

$$\sqrt{1 - \xi} \cdot f' - \alpha_1 f = 0, \quad \xi \leq \xi_2 \quad (5.6b)$$

where  $\xi_2 = 1 - r_b^2 / r_c^2$ . We may take  $r_b$  as the boundary layer thickness or the width of a wake or a jet.

\*Equation (5.3) corresponds to equation (6.13) of Reference 2., but there seems a printing error of equation (6.13) of Reference 2, i.e.,  $f_4$  should be  $f_3$ . Our G's are f's in Reference 2.

The determinantal equation corresponding to the conditions (5.6)

is

$$F_2(c_1, \alpha_1, \alpha_1 R_1) = \begin{vmatrix} f_{11} & f_{21} & f_{41} \\ f'_{11} & f'_{21} & f'_{41} \\ \sqrt{1-\xi_2} \cdot f'_{12} - \alpha_1 f_{12} & \sqrt{1-\xi_2} f'_{22} - \alpha_1 f_{22} & 0 \end{vmatrix} = 0 \quad (5.7)$$

or

$$\frac{f_{41}}{f'_{41}} = \frac{(1-\xi_2)^{\delta/2} G_2 - \alpha_1 G_1}{(1-\xi_2)^{\delta/2} G_4 - \alpha_1 G_3} \quad (5.8)$$

where  $\delta = 0$  for two-dimensional case and  $\delta = 1$  for axisymmetrical case.

## SECTION 6 - EQUIVALENT PRINCIPLE

Since both the fundamental solutions and the determinantal equations for the axisymmetrical flow in the variable  $\xi$  are similar to those of the two-dimensional flow in the variable  $y$ , we may find some qualitative relations between these two cases.

For the two-dimensional flow, the second derivative of the mean velocity profile  $d^2W/dy^2$  plays an important role in the stability problem. We expect that the second derivative  $d^2W/d\xi^2$  of the axisymmetrical flow plays a similar role as  $d^2W/dy^2$  in the corresponding stability problem. For instance, the case of  $d^2W/dy^2 = 0$  should correspond to the case of  $d^2W/d\xi^2 = 0$ . In the two-dimensional case,  $d^2W/dy^2 = 0$  represents the plane Couette flow case which is stable for infinitesimal disturbances at all Reynolds numbers. In the axisymmetrical case,  $d^2W/d\xi^2 = 0$  is the Poiseuille flow in a circular pipe which is also stable for infinitesimal disturbances at all Reynolds numbers.

For the two-dimensional flow problem, if the basic flow has a point of inflection in the flow field, i.e.,  $W''(y_c) = 0$  for  $y_1 < y_c < y_2$ , the flow is very unstable.

The minimum critical Reynolds number of the basic flow with a point of inflection is much smaller than those without a point of inflection. For instance, the two-dimensional wake or jet flows are much more unstable than the boundary layer flow. The corresponding case in axisymmetrical flow should be  $W''(\xi_c) = 0$  for  $\xi_1 > \xi_c > \xi_2$ . Even though  $d^2W/dr^2 = 0$  for  $r = r_c$  where  $r_1 < r_c < r_2$  for an axisymmetrical wake or jet  $d^2W/d\xi^2$  for  $\xi_1 > \xi > \xi_2$ . For instance, for the far downstream wake behind a body of revolution, the basic velocity profile is

$$W(r) = 1 - b \exp(-r^2) \quad (6.1a)$$

or

$$W(\xi) = 1 - b \exp \left[ -r_c^2 (1 - \xi) \right] \quad (6.1b)$$

where  $b$  is a constant in the stability analysis (see Part II). The flow field lies between  $r_1 = 0$  and  $r_2 = \infty$  or  $\xi_1 = 1$  and  $\xi_2 = -\infty$

$$\frac{d^2 W}{dr^2} = 2b(1 - 2r^2) \exp(-r^2) \quad (6.2)$$

Hence, 
$$\frac{d^2 W}{dr^2} = 0 \quad \text{when} \quad r_1 = 0 < r_c = \frac{1}{\sqrt{2}} < r_2 = \infty \quad (6.3)$$

However, 
$$\frac{d^2 W}{d\xi^2} = -br_c^4 \exp[-r_c^2(1 - \xi)] \neq 0 \quad (6.4)$$

when  $1 > \xi > -\infty$

We thus expect that the axisymmetrical wake or jet will be more stable than the corresponding two-dimensional wake or jet because there is no "effective" point of inflection in the mean velocity profile for an axisymmetrical wake or jet, i.e.,  $d^2 W/d\xi^2 \neq 0$  for  $\xi_1 > \xi > \xi_2$ .

The axis of symmetry in the axisymmetrical flow behaves quite different from the axis of symmetry in the two-dimensional flow with symmetrical profile because the boundary conditions for these two cases are different. In the axisymmetrical case, the boundary conditions on the axis of symmetry are the same as those on a solid wall, i.e., both  $f$  and  $f'$  are zero. For the two-dimensional case, only one of the functions  $f$  and  $f'$  vanishes on the axis of symmetry depending on whether the disturbance is symmetrical or antisymmetrical. Hence, the axis of symmetry behaves like a solid wall in the axisymmetrical case but it is not so in the two-dimensional case. As a result, in the stability analysis of an axisymmetrical flow, the viscous solution always takes care of the effect of viscosity no matter whether there is a solid wall or not. We may draw the following

equivalent principles between the two-dimensional and the axisymmetrical cases of stability analysis:

- (a) For flow of finite domain, the stability problem of an axisymmetrical flow corresponds to the stability problem of a two-dimensional flow of unsymmetrical profile when the variable  $\xi$  is used for the variable  $y$ , the two-dimensional variable. The stability equation for both cases is equation (5.3). If  $W(\xi) = W(y)$  we would expect that the critical Reynolds number for these two cases are approximately the same.

Because of the above principle, it is expected that the stability problem of the Poiseuille flow in a circular pipe will be the same as that of plane Couette flow because  $W(\xi) = A + B\xi$  and  $W(y) = A + By$ . Actually we know that both cases are stable to infinitesimal disturbance for all Reynolds numbers.

It should be pointed out that Pretsch<sup>1</sup> did not use the correct determinantal equation (5.3) to investigate the stability of a Poiseuille flow. He arbitrarily discarded the solution  $f_3$  which is large but still finite on the axis of symmetry  $\xi = 1$ . He used then the functions  $f_1$ ,  $f_2$  and  $f_4$  in the stability calculation. He had to discard one of boundary conditions on the axis of symmetry. He used  $f' = 0$  as the boundary conditions as the axis of symmetry. Finally his determinantal equation in our notations is

$$F_3 = \begin{vmatrix} f_{11} & f_{21} & f_{41} \\ f'_{11} & f'_{21} & f'_{41} \\ f'_{12} & f'_{22} & f'_{42} \end{vmatrix} \cong f_4 G_4 - f'_{41} G_2 = 0$$

or

$$\frac{f_{41}}{f'_{41}} = \frac{G_2}{G_4} \quad (6.5)$$

Pretsch found that the Poiseuille flow is stable according to equation (6.5).

- (b) For flow of infinite domain, the stability problem of an axisymmetrical flow always corresponds to the stability problem of a two-dimensional boundary layer flow when the variable  $\xi$  is used for the variable  $y$ . The stability equation for both cases is equation (5.8). If  $W(\xi) = W(y)$ , we would expect that the critical Reynolds numbers for these cases are of the same order of magnitude.

This result is especially interesting in the consideration of the stability of an axisymmetrical wake or jet. We expect that the minimum critical Reynolds number of an axisymmetrical wake or jet would be of the same order of magnitude to that of a two-dimensional boundary layer flow which is much larger than that for a two-dimensional wake or jet. The minimum critical Reynolds number of Blasius boundary layer flow based on the displacement thickness is of the order of 500. For boundary layer flows with pressure gradient,<sup>4</sup> the minimum critical Reynolds number varies from 100 to 1000. On the other hand, the minimum critical Reynolds number of a two-dimensional wake or jet<sup>5</sup> based on the width of the wake or jet is of the order of 10. Hence, we expect that the minimum critical Reynolds number of an axisymmetrical wake or jet should be at least an order of magnitude higher than that of a two-dimensional wake or jet.

Numerical calculations will be made to check the above principles. It should be pointed out that no calculation of the stability of a two-dimensional boundary layer flow of velocity profile similar to equation (6.1) has been carried out. The velocity profile (6.1) may be considered as a uniform velocity  $(1-b)$  superimposed on a variable velocity  $W_b$ , i.e.,

$$W = 1 - b + W_b$$

or

$$W_b = b \left[ 1 - \exp(-r^2) \right] = b \bar{W} \quad (6.6a)$$

Now if we replace  $c$  by  $\bar{c}$  such that

$$b\bar{c} = c - (1-b) \quad (6.7)$$

The stability equation (4.2) holds true for  $\bar{W}$  and  $\bar{c}$  except that the Reynolds number  $R_1$  should be replaced by  $R_{1b}$  which is

$$R_{1b} = R_1 b \quad (6.8)$$

In other words, if we use the actual velocity defect  $bU$  which is the difference between the velocity at the edge of the wake and the velocity on the axis of symmetry as our characteristic velocity, we may use the velocity profile  $\bar{W}$  to investigate the stability of the wake. We shall use this transformation in our investigation of a wake flow in Part II.  $\bar{W}$  is closer to the ordinary boundary layer profile than  $W$ .

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SUMMARY		
<p>The stability equation of an axisymmetrical flow of an incompressible fluid subjected to rotationally-symmetric disturbances is derived and studied. It is shown that the axisymmetrical flow is always stable with respect to disturbances of the tangential velocity component. Hence, the stability equation for axisymmetrical disturbance is then investigated in detail for arbitrary basic velocity profiles. It was found that the stability of an axisymmetrical flow may be reduced approximately to the corresponding problem of two-dimensional flow by proper transformations. The comparison of stability of an axisymmetrical flow with that of a corresponding two-dimensional flow is discussed according to an equivalent principle. It is found that the minimum critical Reynolds number of an axisymmetrical wake, or jet, is of the same order of magnitude as that of a two-dimensional boundary layer flow which is much higher than the value for a two-dimensional wake, or jet.</p>		

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